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# How rare events shape the learning curves of hierarchical data

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## Abstract

1 The learning curves of deep learning methods often behave as a power of the  
2 dataset size. The theoretical understanding of the corresponding exponent yields  
3 fundamental insights about the learning problem. However, it is still limited  
4 to extremely simple datasets and idealised learning scenarios, such as the lazy  
5 regime where the network acts as a kernel method. Recent works study how deep  
6 networks learn synthetic classification tasks generated by probabilistic context-free  
7 grammars: generative processes which model the hierarchical and compositional  
8 structure of language and images. Previous studies assumed composition rules  
9 to be equally likely, leading to non-power-law behavior for classification. In  
10 realistic dataset, instead, some rules may be much rarer than others. By assuming  
11 that the probabilities of these rules follow a Zipf law with exponent  $a$ , we show  
12 that the classification performance of deep neural networks decays as a power  
13  $\alpha = a/(1 + a)$  of the number of training examples, with a large multiplicative  
14 constant that depends on the hierarchical structure of the data.

## 15 1 Introduction

16 The improvement in performance of many machine-learning models with the amount of resources,  
17 including number of model parameters and training examples, has been shown to follow a simple  
18 power-law behaviour across several orders of magnitude [1, 2]. These power laws, known as *neural*  
19 *scaling laws*, are used in practice as a guideline for scaling up resources [3, 4]. Furthermore, they  
20 offer the possibility of explaining complex learning systems with only a few exponents.

21 Among scaling laws, the *learning curve* describes the improvement of test performance with the  
22 number of training examples. A simple approach, based on pure memorisation, leads to power-law  
23 learning curves after assuming Zipf distributed data [5, 6, 7]. In practice, however, data are rarely  
24 seen twice as this viewpoint assumes. Alternatively, power-law learning curves can occur in simple  
25 machine learning methods such as kernel regression. It occurs if the coefficient of the true function in  
26 the kernel basis decay as a power-law of rank, as expected from bounds [8, 9] or direct estimation of  
27 exponents [10, 11, 12, 13, 14, 15, 16], a hypothesis confirmed empirically [10, 11, 12]. However,  
28 these approaches are restricted to kernel-based approximations of deep learning methods, whose  
29 limited power cannot explain the successes of modern, large language and vision models.

30 In this respect, recent studies have identified hierarchical generative models such as probabilistic  
31 context-free grammars as model datasets that explain the difference in performance and data efficiency  
32 between deep learning methods and kernels or other shallow methods, while still simple enough to  
33 allow for some analytical understanding [17, 18, 19, 20, 21, 22, 23]. For these tasks, the input data  
34 are generated from their class labels according to a hierarchy of production rules, mapping high-level  
35 features to tuples of lower-level features. If productions rules are evenly distributed, learning curves  
36 are not power-law for classification problems [19], although they are for next-token prediction [22].

37 In this work, we combine one such data model—the Random Hierarchy Model (RHM) of [19]—with  
 38 the hypothesis that the production rules of real datasets (e.g. the words in a text corpus) are Zipf  
 39 distributed. For classification, we find that the performance of deep neural networks decays as a  
 40 power  $\alpha = a/(1 + a)$  of the number of training examples, with a large multiplicative constant that  
 41 depends on the hierarchical structure of the data.

## 42 2 Notation and setup

43 **Hierarchical generative model.** We consider synthetic datasets that model the hierarchical and  
 44 compositional structure of real data such as images and text. These datasets are generated via a  
 45 probabilistic context-free grammar (PCFG) [24]: a collection of symbols and rules that prescribe how  
 46 to generate input data starting from their label. The PCFGs we consider consist of

- 47 •  $L$  finite vocabularies of hidden (nonterminal) symbols  $(\mathcal{V}_\ell)_{\ell=1,\dots,L}$ ;
- 48 • A finite vocabulary of observable (terminal) symbols  $\mathcal{V} \equiv \mathcal{V}_0$ ;
- 49 •  $L$  sets of *production rules* describing how one symbols of  $\mathcal{V}_\ell$  generates a tuple of (lower-  
 50 level) symbols of  $\mathcal{V}_{\ell-1}$ , for  $\ell = 1, \dots, L$ ,

$$\mu^{(\ell)} \rightarrow \mu_1^{(\ell-1)}, \dots, \mu_s^{(\ell-1)}, \quad \text{for } \mu^{(\ell)} \in \mathcal{V}_\ell, \mu_i^{(\ell-1)} \in \mathcal{V}_{\ell-1}, \quad (1)$$

51 for some integer size  $s \geq 1$ . We further assume that *i*) the label set  $\mathcal{V}_L$  has size  $n_c$  and all the other  
 52 vocabularies have size  $v$ ; *ii*) Each hidden symbol of level  $\ell = 1, \dots, L$  enters in  $m$  distinct production  
 53 rules and each of these rules can be picked with probability  $f_i^{(\ell)}$ , with  $i = 1, \dots, m$  and  $\sum_i f_i^{(\ell)} = 1$ ;  
 54 *iii*) two different high-level symbols cannot generate the same lower-level  $s$ -tuple, i.e. low-level  
 55  $s$ -tuples determine the corresponding higher-level symbol *unambiguously*.

56 In the Random Hierarchy Model (RHM) of [19], the production rules are sampled uniformly among  
 57 all the possible sets of rules compatible with the constraints above, and their probabilities  $f_i^{(\ell)}$  are  
 58 set to  $1/m$  for all  $i$ 's and  $\ell$ 's. Here, mimicking the power-law distribution of word frequencies [25],  
 59 we set the production rule distribution to be uniform in all but one layer  $\ell$ , where it follows a Zipf  
 60 law [5, 6],  $f_k^{(\ell)} \propto k^{-(1+a)}$ .

61 Once the model is specified, input data are generated by picking a class label  $y$  (or level- $L$  symbol)  
 62 uniformly at random, then picking a production rule emanating from that label and replacing the  
 63 label with the right-hand side of the production rule. Repeating the process  $L$  times yields the  
 64 input sequence  $x = (x_1, \dots, x_d)$ , with  $d = s^L$ . Each feature  $x_i$  is represented as a  $v$ -dimensional  
 65 one-hot vector  $(x_{i,\mu})_{\mu=1,\dots,v}$ , with  $x_{i,\mu} = 1$  if  $x_i$  encodes for the  $\mu$ -th element of the vocabulary and 0  
 66 otherwise. The probability of the input conditioned on the label,  $\mathbb{P}\{X_1 = x_1, \dots, X_d = x_d | Y = y\}$ ,  
 67 is given by multiplying the probabilities of all the production rules involved.

68 **Learning Setup.** We focus on deep convolutional networks (CNNs) trained for classification by  
 69 gradient descent over the empirical cross-entropy loss,

$$\mathcal{L}(\mathcal{X}_P) = -\frac{1}{P} \sum_{(\mathbf{x}, y) \in \mathcal{X}_P} \log(p_\theta(y|x_1, \dots, x_d)), \quad (2)$$

70 where  $\mathcal{X}_P$  is a set of  $P$  training examples drawn from the joint input-label distribution and  $p_\theta$  denotes  
 71 the parametric approximation of the label probability,

$$p_\theta(y|x_1, \dots, x_d) \approx \mathbb{P}\{Y = y | X_1 = x_1, \dots, X_d = x_d\}. \quad (3)$$

72 Numerical experiments are performed in PyTorch [26], with the code attached as Supplemental  
 73 Material. Details of the machine learning models, training hyperparameters and computer resources  
 74 are presented in App.A.

## 75 3 Input-label correlations and sample complexity

76 **Correlations versus sampling noise.** The analysis of [19] shows that, in the uniform production  
 77 rules case, the sample complexity of deep neural networks trained on RHM data is controlled by the

78 correlations between the class label and the  $s$ -tuples of low-level features. Indeed, these correlations  
 79 can be used to group together tuples corresponding to the same higher-level variable, allowing to  
 80 learn the generative model bottom up. This correlation is measured via the probability for a datum to  
 81 belong to class  $y$  conditioned on displaying the  $s$ -tuple  $\boldsymbol{\mu}$  in the  $j$ -th input patch,

$$p_j^L(y|\boldsymbol{\mu}) := \mathbb{P}\{Y = y | X_{(j-1)s+1} = \mu_1, \dots, X_{js} = \mu_s\}_L, \quad (4)$$

82 with the subscript  $L$  indicating explicitly the depth of the generative model. As shown in [19], these  
 83 correlations can be thought of as random variables over different realisations of the RHM, with mean  
 84  $1/n_c$  and variance  $v/(n_c^2 m^L)$ . The mean coincides with the uniform probability over classes, thus it  
 85 is uninformative. Removing the mean results in a ‘signal’ that can be used to solve the task. However,  
 86 when measuring correlations from a set of  $P$  training examples, these are affected by a sampling  
 87 noise of zero mean and variance  $v/(n_c P)$ . Comparing the sizes of noise and signal results in the  
 88 sample complexity  $P^* = n_c m^L$  actually observed.

### 89 3.1 Nonuniform production rules at the bottom layer

90 Rare productions rules will contribute little to correlations, thus a larger training set will be required  
 91 to detect their effects and learn these rare rules. Assuming that these rules are learnt precisely when  
 92 their effect on the correlation becomes detectable offers a perspective on the learning curve, which  
 93 we will examine further below.

94 The probability of a low-level tuple conditioned on the class reads

$$\mathbb{P}\{\mathbf{X}_j = \boldsymbol{\mu} | Y = y\}_L = f_{k(\boldsymbol{\mu})}^{(1)} \mathbb{P}\{X_j = \mu_1(\boldsymbol{\mu}) | Y = y\}_{L-1}, \quad (5)$$

95 where  $f_{k(\boldsymbol{\mu})}^{(1)}$  is the probability of the unique production rule that generates  $\boldsymbol{\mu}$ , and  $\mu_1(\boldsymbol{\mu})$  the unique  
 96 level-1 features that generates  $\boldsymbol{\mu}$  via that production rule. Summing over  $y$  yields a similar result for  
 97 the probability of  $\boldsymbol{\mu}$ :  $\mathbb{P}\{\mathbf{X}_j = \boldsymbol{\mu}\}_L = f_{k(\boldsymbol{\mu})}^{(1)} \mathbb{P}\{X_j = \mu_1(\boldsymbol{\mu})\}_{L-1}$ .

98 Since  $p_j^L(y|\boldsymbol{\mu})$  is proportional to the ratio  $\mathbb{P}\{X_j = \boldsymbol{\mu} | Y = y\}_L / \mathbb{P}\{\mathbf{X}_j = \boldsymbol{\mu}\}_L$ , it is independent  
 99 of  $f_{k(\boldsymbol{\mu})}^{(1)}$  and identical to the model with uniform production rules. However, the sampling noise is  
 100 affected by the probability of the production rules, as the number of data with  $\mathbf{X}_j = \boldsymbol{\mu}$  is proportional  
 101 to  $f_{k(\boldsymbol{\mu})}^{(1)}$ . With respect to the case where all low-level tuples have the same probability, this effect  
 102 is equivalent to replacing  $P$  by  $P f_{k(\boldsymbol{\mu})}^{(1)} m$ , which recovers in the uniform case  $f_{k(\boldsymbol{\mu})}^{(1)} = 1/m$ . Thus  
 103 it results in a sampling noise of variance  $v/(n_c f_{k(\boldsymbol{\mu})}^{(1)} P)$ . Therefore, the sample size necessary to  
 104 resolve the correlations of the tuple  $\boldsymbol{\mu}$  with the label is  $P^*(\boldsymbol{\mu}) = n_c m^{L-1} / (f_{k(\boldsymbol{\mu})}^{(1)})$ .

105 Ranking all the low-level tuples by the probability of the corresponding production rules yields  
 106 a sequence of  $m$  sample complexities  $P_k^* = n_c m^{L-1} / (f_k^{(1)})$ . To estimate the learning curve, we  
 107 assume that, when  $P > P_k^*$ , the model can correctly classify data consisting of tuples with probability  
 108 higher than  $f_k$ . Then the model could correctly classify if and only if all  $s^{L-1}$  input layer patches are  
 109 resolvable. The resulting test error, defined as the probability of misclassification, reads

$$\varepsilon(P) = 1 - \left( \sum_{k | P_k^* < P} f_k^{(1)} \right)^{s^{L-1}}. \quad (6)$$

110 When  $P \gg P_1^* \simeq n_c m^{L-1}$ , this expression implies, as shown in App.B,

$$\varepsilon(P) \simeq s^{L-1} \left( \frac{P}{n_c m^{L-1}} \right)^{-a/(1+a)}. \quad (7)$$

111 These arguments are extended to nonuniform production rules in an arbitrary layer in App.C.

## 112 4 Empirical scaling laws of deep CNNs

113 We confirm the results of Eqs. 6,7 with the measurements of learning curves of CNNs learning the  
 114 RHM for (i) varying Zipf exponent  $a$  in Fig.1.A, (ii) varying the layer where the production rules  
 115 are following a Zipf law in Fig.1.B, (iii) varying the number of production rule per symbol  $m$  in  
 116 Fig.1.C,D. The comparisons are excellent in all cases. Further evidence exploring more parameters  
 117 are given in App.D.

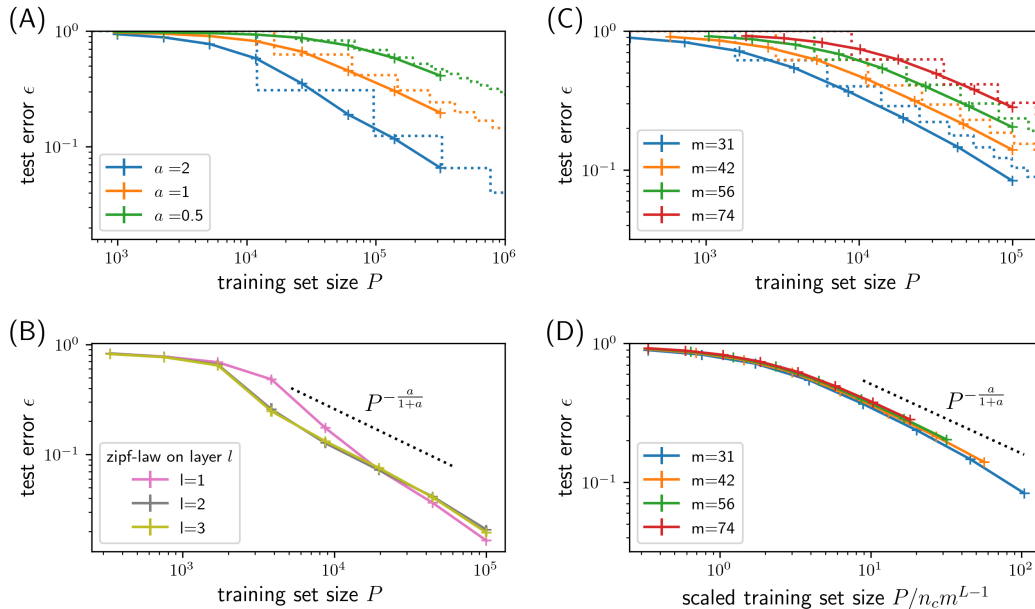


Figure 1: **(A)**: Learning curves of CNN trained on  $L = 2, m = 100$ , zipf-law on input layer ( $l = 1$ ) varying zipf-law exponent  $a$ . Dotted lines are predictions from Eq.6. **(B)**: Learning curves of CNN trained on  $L = 3, m = 10, a = 2$ , varying the layer implementing Zipf-law. Black dotted line is our scaling exponent from Eq.7. **(C),(D)**: Learning curves of CNN trained on  $L = 2, a = 1$ , varying  $m$ . (D) is same plot with (C) except for rescaling x-axis. Dotted lines in (C) is theoretical prediction from Eq.6, Black dotted line in (D) is from Eq.7.

## 118 5 Conclusions

119 In the simplest context-free grammars where the generative tree is fixed, and production rules are  
 120 uniform and random, the learning curve for classification of the top node is not power-law. Instead,  
 121 it is characterized by a single sample scale, polynomial in the number of rules  $m$  and exponential  
 122 in the depth  $L$  of the generative model. We have shown that if production rules are not uniform  
 123 but power-law distributed with some exponent  $a$ , then the learning curve inherits that property and  
 124 decays as  $m^{(L-1)a/(a+1)} P^{-a/(a+1)}$ . The exponent  $a/(a+1)$  is also found in elementary toy models  
 125 of Zipf law learning [3, 4]. Yet, in our case, as in real data, this behaviour is not simply caused  
 126 by pure memorization, as a datum is never seen twice for large  $L$ <sup>1</sup>. Interestingly, we find that the  
 127 pre-factor in front of the power-law behaviour of the learning curve can be very large, and depends  
 128 on the combinatorial nature of the problem.

129 In real data, we expect other factors to shape scaling law exponents, including the shape of the  
 130 generative tree. For example, the tree depth  $L$  does not need to be single-valued, as would occur e.g.  
 131 for images acquired with different zoom levels. For next-token prediction, other effects associated  
 132 with long-range correlations, present already for uniform production rules and fixed tree depth, also  
 133 affect learning curve exponents [22]. It is plausible that among all these effects, the one leading to  
 134 the smallest exponent would dominate the learning curve. An interesting question for the future is  
 135 to understand which contribution dominates empirically in which datasets, and estimate parameters  
 136 characterizing the structure of real data.

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<sup>1</sup>The total number of data these models generate is doubly exponential in  $L$ , while the sample complexity simply exponential in  $L$ .

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## 205 **A Methods**

### 206 **A.1 RHM implementation**

207 The code implementing the RHM is available online at <https://github.com/pcsl-epfl/hierarchy-learning/blob/master/datasets/hierarchical.py>. The inputs sampled from the RHM are represented  
208 as a one-hot encoding of low-level features so that each input consists of  $s$   $L$  pixels and  $v$  channels  
209 (size  $s L \times v$ ). The input pixels are whitened over channels, i.e., each pixel has zero mean and unit  
210 variance over the channels.  
211

### 212 **A.2 Deep CNNs**

213 The deep CNNs we consider are made by stacking standard convolutional layers. To tailor the  
214 network to the structure of the data generative model, we fix both the stride and filter size of these  
215 layers to  $s$ . Since each layer reduces the spatial dimensionality by a factor  $s$ , the input size  $d$  must  
216 be an integer power of  $s$  and the CNNs depth equals  $\log d / \log s$ . We use the Rectified Linear Unit  
217 (ReLU)  $(x) = \max(0, x)$  as activation function.

### 218 **A.3 Trainig Procedure**

219 Training is performed within the PyTorch deep learning framework [67]. Neural networks are trained  
220 on  $P$  training points sampled uniformly at random from the RHM data, using Gradient Descent on  
221 the cross-entropy loss. Learning rate is initialised to 1 and follows a cosine annealing schedule over  
222  $3 \times 10^4$  epochs. Training stops when the training loss reaches  $10^{-2}$ . The performance of the trained  
223 models is measured as the classification error on a test set. The size of the test set is set to  $2 \times 10^6$ .  
224 Reported results for a given value of RHM parameters are averaged over 10 jointly different instances  
225 of the RHM and network initialization for  $m \leq 20$ , and 3 instances for  $m > 20$ .

## 226 **B Asymptotics of the learning curve $\varepsilon(P)$**

227 Definition of  $g(P)$ :

228 We define  $g(P)$  as the proportion of contributions from the first  $k'$  terms:

$$g(P) = \sum_{k=1}^{k'} f_k^{(1)} = \frac{\sum_{k=1}^{k'} k^{-1-a}}{\sum_{k=1}^m k^{-1-a}}, \quad (8)$$

229 where  $k'$  is the largest integer  $k$  such that  $f_k^{(1)} > n_c m^{L-1} / P$ .

230 Derivation of  $k'$ :

231 Starting with the condition  $P_k^* < P$ :

232

233 Substituting  $f_k^{(1)} = \frac{k^{-1-a}}{\sum_{j=1}^m j^{-1-a}}$ , this inequality simplifies to:

234

235 Rearranging gives:

$$k^{-1-a} > \frac{n_c m^{L-1}}{P} \cdot \sum_{j=1}^m j^{-1-a}, \quad (9)$$

$$k' \sim \left( \frac{P}{n_c m^{L-1}} \right)^{\frac{1}{1+a}}. \quad (10)$$

236 Approximation of  $g(P)$ :

237 Now, substituting  $k'$  into the expression for  $g(P)$ :

$$g(P) = \frac{\sum_{k=1}^{k'} k^{-1-a}}{\sum_{k=1}^m k^{-1-a}}. \quad (11)$$

238 Approximation Using the Euler-Maclaurin Formula:

239 To approximate these sums, we use the Euler-Maclaurin formula, which relates sums to integrals:

$$\sum_{k=a}^b f(k) \approx \int_a^b f(x) dx + \frac{f(a) + f(b)}{2} + \sum_{j=1}^n \frac{B_{2j}}{(2j)!} f^{(2j-1)}(x) \Big|_a^b,$$

240 where  $B_{2j}$  are Bernoulli numbers. For large  $b$ , the correction terms diminish, allowing the sum to be  
241 closely approximated by the integral.

242 1. Numerator of  $g(P)$ :

243 Approximating the sum from 1 to  $k'$ :

$$\sum_{k=1}^{k'} k^{-1-a} \approx \int_1^{k'} x^{-1-a} dx + \frac{k'^{-1-a} + 1^{-1-a}}{2} + \text{higher-order corrections}, \quad (12)$$

$$\approx \int_1^{k'} x^{-1-a} dx, \quad (\text{neglecting boundary terms for large } k'). \quad (13)$$

244 The boundary term  $\frac{1^{-1-a}}{2} = \frac{1}{2}$  is negligible relative to the integral, especially when  $a > 0$ , because  
245 the integral scales with  $k'$ , which grows large as  $P$  increases.

246 2. Denominator of  $g(P)$ :

247 For the sum from 1 to  $m$ , we approximate by extending the upper limit to infinity:

$$\sum_{k=1}^m k^{-1-a} \approx \int_1^m x^{-1-a} dx + \frac{m^{-1-a} + 1^{-1-a}}{2} + \text{correction terms}, \quad (14)$$

$$\approx \int_1^{\infty} x^{-1-a} dx, \quad (\text{extending to infinity is valid as } m \text{ is large}). \quad (15)$$

248 For large  $m$ , the tail of the integral beyond  $m$  contributes negligibly, as  $x^{-1-a}$  decays rapidly when  
249  $a > 0$ . The impact on the result is minimal, making the approximation practically accurate.

250 Evaluating the Integrals:

$$\int_1^{k'} x^{-1-a} dx = \left[ \frac{x^{-a}}{-a} \right]_1^{k'} = \frac{1}{a} (1 - (k')^{-a}), \quad (16)$$

$$\int_1^{\infty} x^{-1-a} dx = \left[ \frac{x^{-a}}{-a} \right]_1^{\infty} = \frac{1}{a}. \quad (17)$$

251 Since  $m^{-a} \rightarrow 0$  for large  $m$ , the integral in the denominator simplifies to  $\frac{1}{a}$ .

252 Substituting into  $g(P)$ :



$$g(P) \approx \frac{\frac{1}{a} [1 - (k')^{-a}]}{\frac{1}{a}} \quad (18)$$

$$= 1 - (k')^{-a} \quad (19)$$

$$= 1 - c \left( \frac{P}{n_c m^{L-1}} \right)^{-\frac{a}{1+a}}, \quad \text{for some constant } c. \quad (20)$$

253 Approximation of  $\varepsilon(P)$ :

254 Substituting  $g(P)$  into the expression for  $\varepsilon(P)$ :

$$\varepsilon(P) = 1 - \{g(P)\}^{s^{L-1}} \quad (21)$$

$$\approx 1 - \left[ 1 - c \left( \frac{P}{n_c m^{L-1}} \right)^{-\frac{a}{1+a}} \right]^{s^{L-1}}. \quad (22)$$

255 Using a Taylor expansion around  $g(P) \approx 1$  for large  $P$ :

$$\approx s^{L-1} \cdot c \left( \frac{P}{n_c m^{L-1}} \right)^{-\frac{a}{1+a}}, \quad \text{for } P \gg n_c m^{L-1}. \quad (23)$$

$$\sim \left( \frac{P}{n_c m^{L-1}} \right)^{-\frac{a}{1+a}} \quad (24)$$

256 Thus, the final approximation is:

$$\varepsilon(P) \sim \left( \frac{P}{n_c m^{L-1}} \right)^{-\frac{a}{1+a}}. \quad (25)$$

## 257 C Nonuniform production rules on an arbitrary layer

258 When the nonuniform distribution of production rules affects an arbitrary layer  $\ell \neq 1$ , the probabilities  
 259 of low-level tuples can be decomposed as sums of conditional probabilities over a specific choice  
 260 of production rules. As a result, the conditional class probability  $p_j^L(y|\mu\boldsymbol{\mu})$  can be written as a sum  
 261 of contributions due to production rules of a given probability  $f_k^{(\ell)}$ . Let us assume that the correct  
 262 classification of data containing production rules with probability  $f_k^{(\ell)}$  requires the accurate resolution  
 263 of the corresponding contribution to the label-tuples correlations. Then, we can apply again the  
 264 argument of the previous section, and derive Eq. 6 for the behaviour of the test error.

## 265 D More empirical results

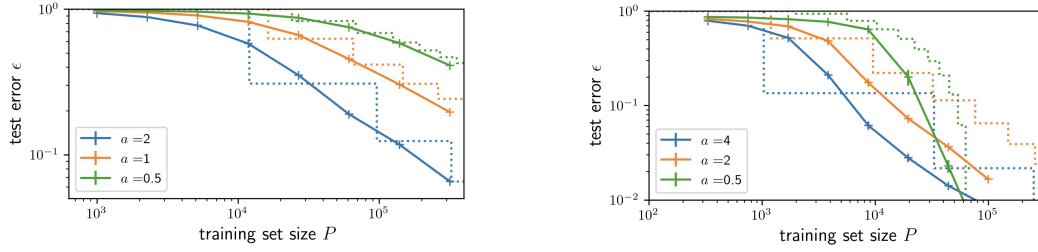


Figure 2: **Left:** Learning curves of CNN trained on  $L = 2, m = 100$ . Zipf-law on input layer( $l = 1$ ) varying Zipf-law exponent  $a$ . Dotted lines are predictions from Eq.6. **Right:** Learning curves of CNN trained on  $L = 3, m = 10$  (Top) Zipf-law on input layer( $l = 1$ ) varying Zipf-law exponent  $a$ . Dotted lines are from our theoretical prediction.

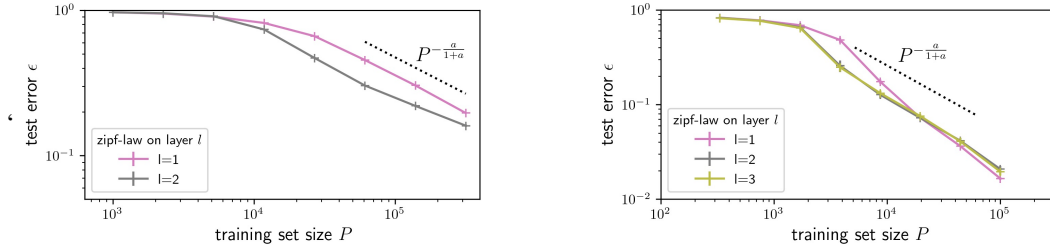


Figure 3: **Left:** Learning curves of CNN trained on  $L = 2, m = 100$ . Zipf-law with exponent  $a = 1$  varying which layer to implement it. **Right:** Learning curves of CNN trained on  $L = 3, m = 10$ . Zipf-law with exponent  $a = 2$  varying which layer to implement it.

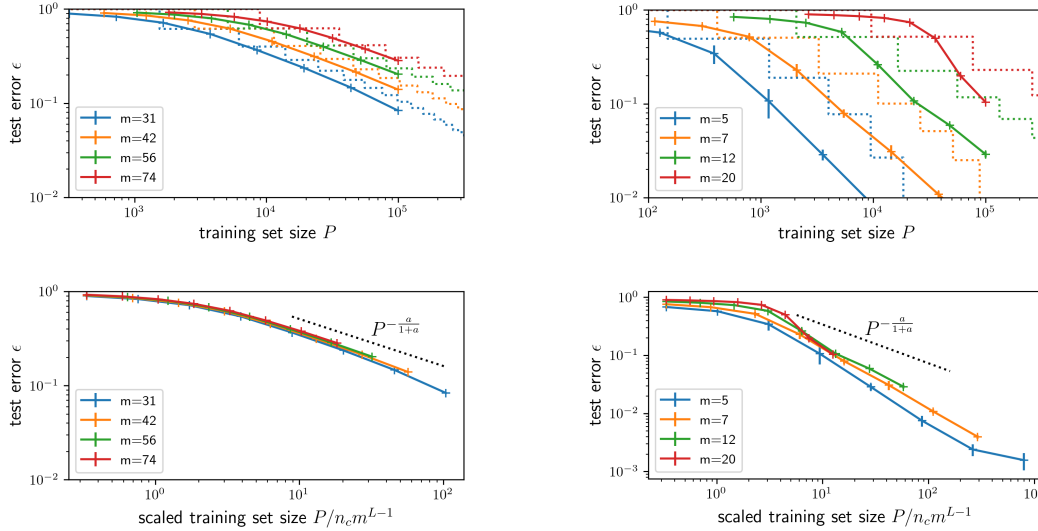


Figure 4: **Left:** Learning curves of CNN trained on  $n_c = v = m, L = 2, s = 2$ . (Top) Zipf-law on last layer with  $a = 1$  and varying  $m$ . Dotted lines are from our theoretical prediction Eq.6. (Bottom) same plot with x-axis was scaled by dividing  $n_c m^{L-1}$  so that curves collapse, as expected from Eq.7. **Right:** Learning curves of CNN trained on  $n_c = v = m, L = 3, s = 2$ . (Top) Zipf-law on last layer with  $a = 2$  and varying  $m$ . Dotted lines are from our theoretical prediction Eq.6. (Bottom) same plot with x-axis rescaled so that curves collapse. Black dotted line is  $P^{-\alpha} = P^{-\frac{\alpha}{1+\alpha}}$  from Eq.7.